Group-Subgroup Relations I. General Considerations

I. Subgroups: index, coset decomposition and normal subgroups

II. Conjugate elements and conjugate subgroups, factor groups

III. Normalizers

DEFINITION. The symmmetry operations of an object constitute its symmetry group.

DEFINITION. A group is a set $G = \{e, g_1, g_2, g_3 \ldots\}$ together with a product $\circ$, such that

i) G is "closed under $\circ$": if g_1 and g_2 are any two members of G then so are $g_1 \circ g_2$ and $g_2 \circ g_1$;

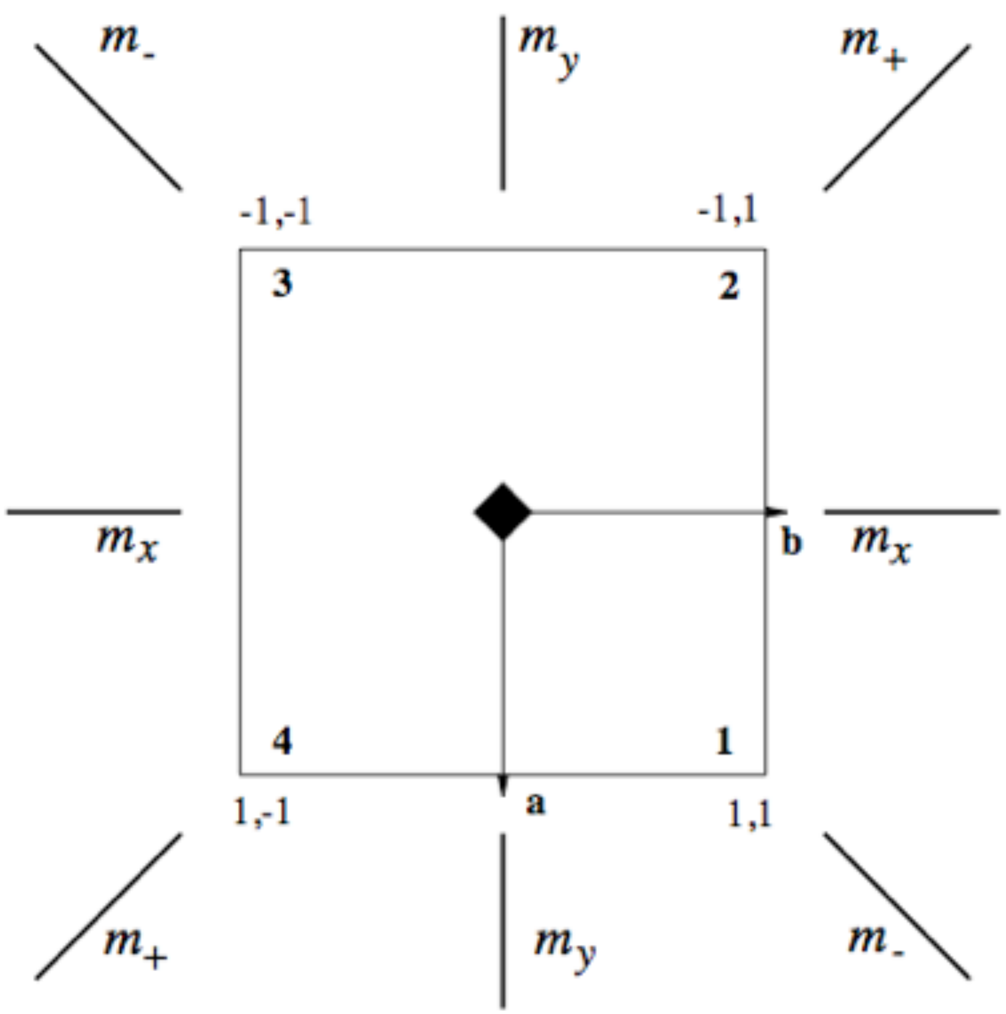
ii) G contains an identity e: for any g in G, $e \circ g = g \circ e = g$;

iii) $\circ$ is associative: $(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$;

iv) Each g in G has an inverse g^{-1} that is also in G: $g \circ g^{-1} = g^{-1} \circ g = e$.

If every element of G can can be written as a product of elements of some subset $\{g_1, \ldots, g_k\}$, then this subset generates G.

Find a set of two generators for the symmetry group of the square.



apply this element first

	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

Multiplication table of $4mm$

Subgroups: Some basic results (summary)

Subgroup H < G

1. $H=\{e,h_1,h_2,\ldots,h_k\} \subset G$
2. H satisfies the group axioms of G

Proper subgroups H < G, and

trivial subgroup: {e}, G

Index of the subgroup H in G: [i]=|G|/|H|

(order of G)/(order of H)

Maximal subgroup H of G

NO subgroup Z exists such that:

H < Z < G

Supergroups: Some basic results (summary)

Supergroup G>H

$H=\{e,h_1,h_2,\ldots,h_k\} \subset G$

Proper supergroups G>H, and
trivial supergroup: H

Index of the group H in supergroup G: $[i]=|G|/|H|$
(order of G)/(order of H)

Minimal supergroups G of H

NO subgroup Z exists such that:
$H < Z < G$

Coset decomposition G:H

Group-subgroup pair H < G

left coset decomposition $G = H + g_2H + \ldots + g_mH,\ g_i \notin H,$ m=index of H in G

right coset decomposition $G = H + Hg_2 + \ldots + Hg_m,\ g_i \notin H$ m=index of H in G

Coset decomposition-properties

(i) $g_iH \cap g_jH = \{\varnothing\}$, if $g_i \notin g_jH$

(ii) $|g_iH| = |H|$

(iii) $g_iH = g_jH,\ g_i \in g_jH$

Coset decomposition G:H

Normal subgroups

$Hg_j = g_jH$, for all $g_j = 1, ..., [i]$

Theorem of Lagrange

group G of order |G|
subgroup H<G of order |H|

then

|H| is a divisor of |G|
and [i]=|G:H|

Corollary

The order *k* of any element of G, $g^k = e$, is a divisor of |G|

Conjugate elements

Conjugate elements

$g_i \sim g_k$ if $\exists$ g: $g^{-1} g_i g = g_k$,
where g, g_i, g_k, $\in$ G

Classes of conjugate elements

$L(g_i)=\{g_j |\ g^{-1} g_i g = g_j,\ g \in G\}$

Conjugation-properties

(i) $L(g_i) \cap L(g_j) = \{\varnothing\}$, if $g_i \notin L(g_j)$

(ii) $|L(g_i)|$ is a divisor of $|G|$ (iii) $L(e)=\{e\}$

(iv) if $g_i, g_j \in L$, then $(g_i)^k=(g_j)^k= e$

Conjugate subgroups

Conjugate subgroups Let $H_1 < G$, $H_2 < G$

then, $H_1 \sim H_2$, if $\exists\, g \in G$: $g^{-1} H_1 g = H_2$

(i) Classes of conjugate subgroups: $L(H)$

(ii) If $H_1 \sim H_2$, then $H_1 \cong H_2$

(iii) $|L(H)|$ is a divisor of $|G|/|H|$

Normal subgroup

$H \triangleleft G$, if $g^{-1} H g = H$, for $\forall g \in G$

Factor group

product of sets: $G=\{e, g_2, \ldots, g_p\}$

$$\begin{cases} K_j=\{g_{j1}, g_{j2}, \ldots, g_{jn}\} \\ K_k=\{g_{k1}, g_{k2}, \ldots, g_{km}\} \end{cases}$$

$K_j\, K_k=\{\, g_{jp}g_{kq}=g_r \mid g_{jp} \in K_j,\, g_{kq} \in K_k\}$

Each element g_r is taken only once in the product $K_j\, K_k$

factor group G/H:

$H \triangleleft G$

$G=H+g_2H+\ldots+g_mH,\ g_i \notin H,$

$G/H=\{H, g_2H, \ldots, g_mH\}$

group axioms:

(i) $(g_iH)(g_jH) = g_{ij}H$

(ii) $(g_iH)H = H(g_iH) = g_iH$

(iii) $(g_iH)^{-1} = (g_i^{-1})H$

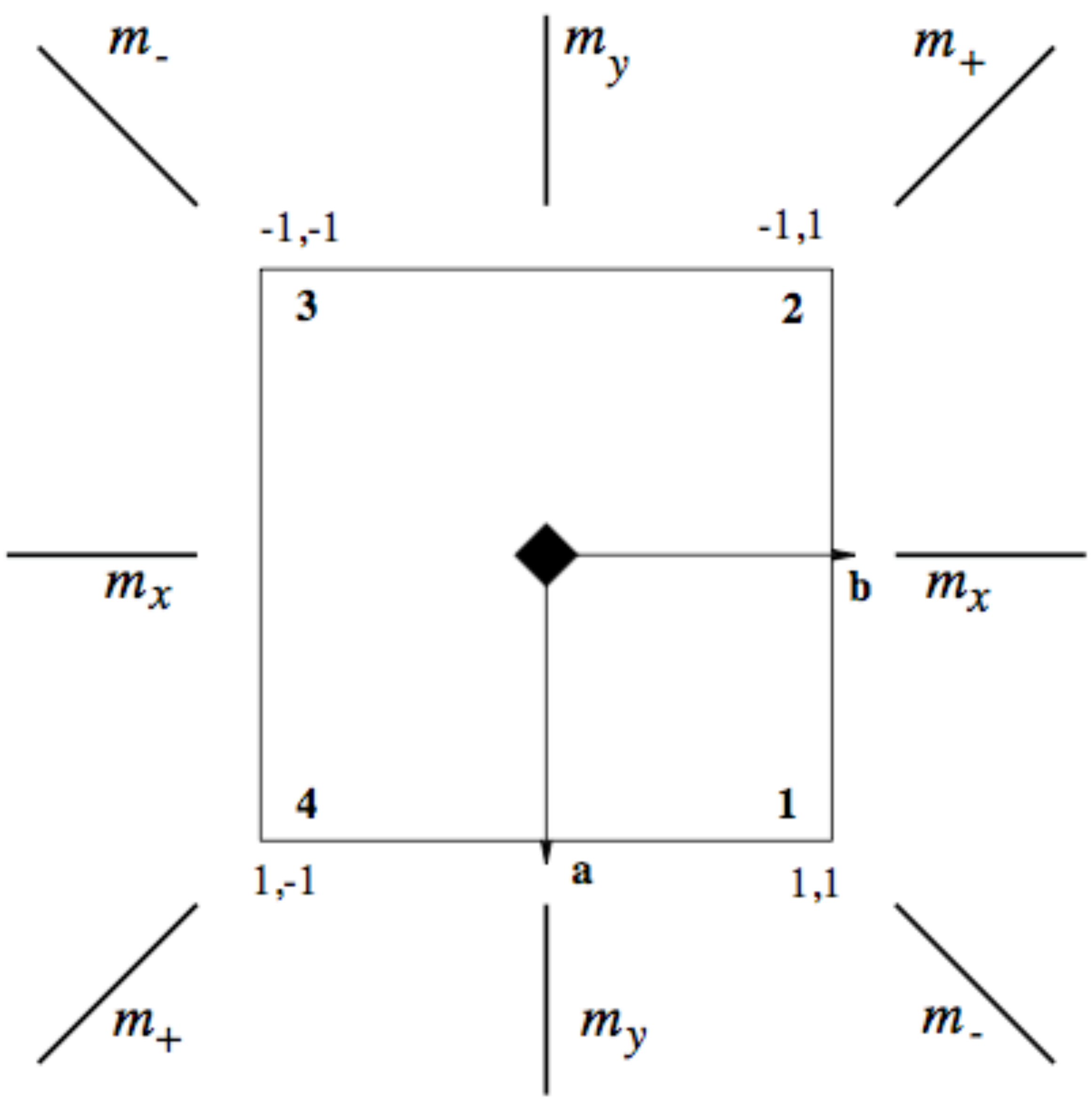
EXERCISES

Problem 3.1

Demonstrate that H is always a normal subgroup if |G:H|=2.

Problem 3.2

The group of the square and its subgroups



	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
1	1	2	4	4^{-1}	m_x	m_+	m_y	m_-
2	2	1	4^{-1}	4	m_y	m_-	m_x	m_+
4	4	4^{-1}	2	1	m_+	m_y	m_-	m_x
4^{-1}	4^{-1}	4	1	2	m_-	m_x	m_+	m_y
m_x	m_x	m_y	m_-	m_+	1	4^{-1}	2	4
m_+	m_+	m_-	m_x	m_y	4	1	4^{-1}	2
m_y	m_y	m_x	m_+	m_-	2	4	1	4^{-1}
m_-	m_-	m_+	m_y	m_x	4^{-1}	2	4	1

Multiplication table of $4mm$

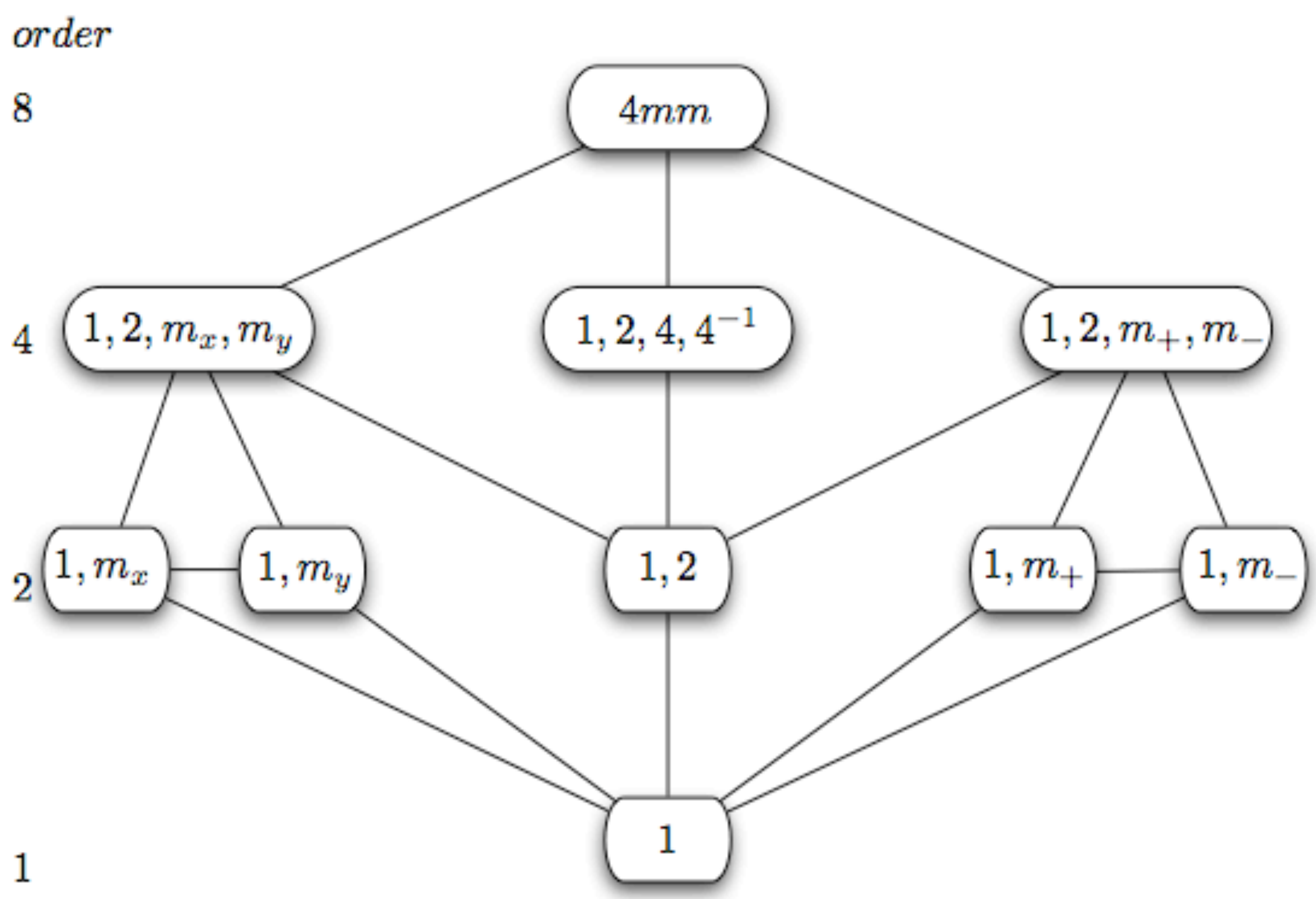
Problem 3.1

SOLUTION

(i) Classes of conjugate elements

$\{e\}, \{4, 4^{-1}\}, \{2\}, \{m_x, m_y\}, \{m_+, m_-\}$

(ii) Group-subgroup diagram



Problem 3.3

Consider the normal subgroup
{e,2} of 4mm, of index 4.

(i) Coset decomposition 4mm: {e,2}

(ii) Show that the cosets of the decomposition 4mm:{e,2}
fulfil the group axioms and form a factor group

(iii) Multiplication table of the factor group

(iv) A crystallographic point group
isomorphic to the factor group?

Problem 3.3

SOLUTION

(i) coset decomposition

$\{e,2\}$, $\{4,4^{-1}\}$, $\{m_x,m_y\}$, $\{m_+,m_-\}$

E A B C

(ii) factor group and multiplication table

	E	A	B	C
E	E	A	B	C
A	A	E	C	B
B	B	C	E	A
C	C	B	A	E

Multiplication table of the Vierergruppe

Example: 222

Normalizer of H in G

Normal subgroup

$H \triangleleft G$, if $g^{-1} H g = H$, for $\forall g \in G$

Normalizer of H in G, H<G

$N_G(H) = \{g \in G, \text{ if } g^{-1} H g = H\}$

$G \geq N_G(H) \geq H$

What is the normalizer $N_G(H)$ if $H \triangleleft G$?

Problem 3.4

Consider the group 4mm and its subgroups of index 4.
Determine their normalizers in 4mm.

Problem 3.5

Plane point groups

(1) **Consider the following 10 figures of the symmetry elements and the general positions of the plane point groups.**

   a) **Determine the order of the point groups and arrange them vertically by descending point-group orders (*i.e.* the point group of highest order at the top, and that of lowest order at the bottom).**

   b) **Determine the complete group-subgroup graph for all plane point groups.**

(2) **Consider the point group $2mm$. Determine its maximal subgroups, its minimal supergroups and the corresponding indices.**

Problem 3.5

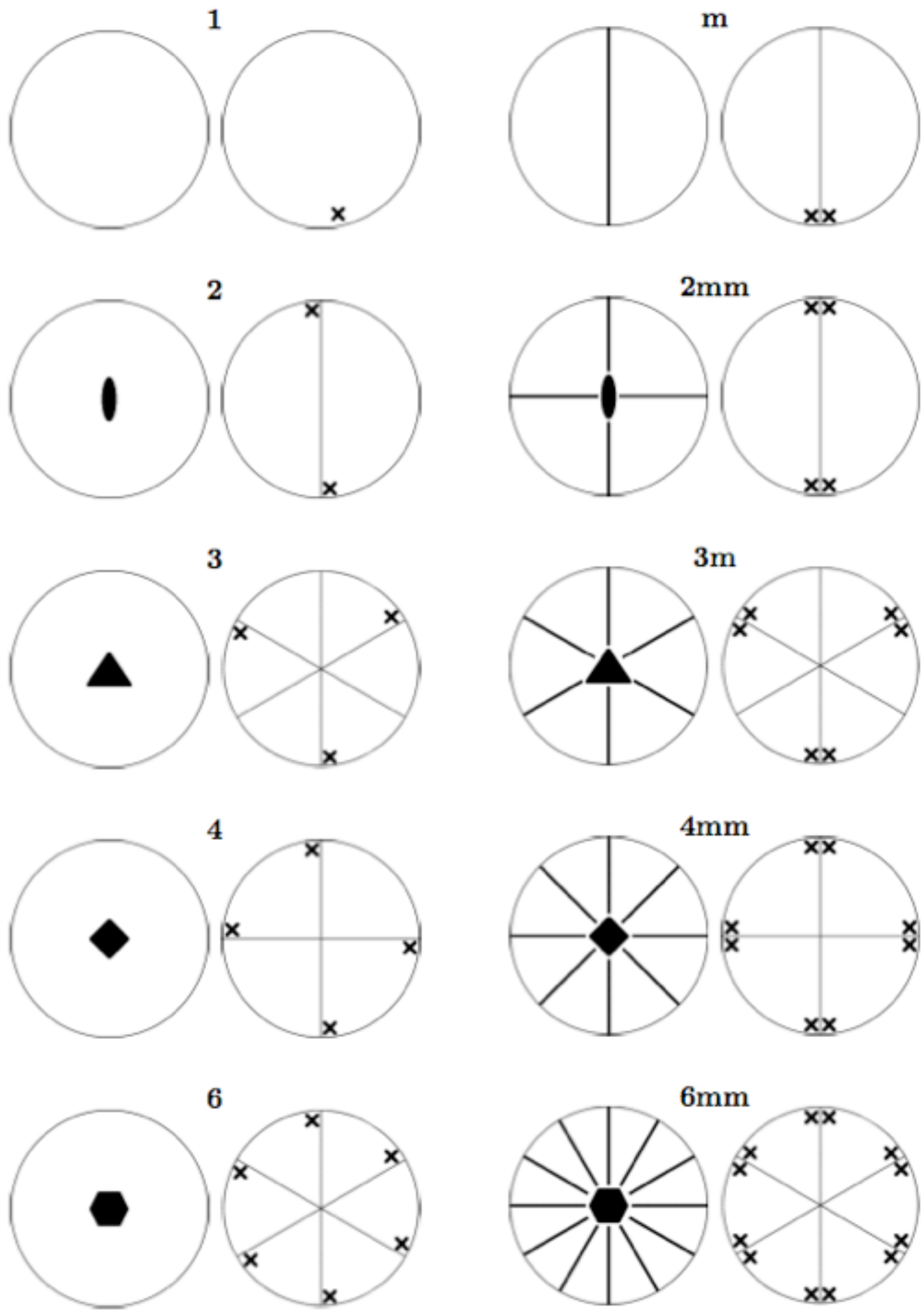
Plane point groups

(3) a) Determine all subgroups of the crystal class $\frac{2}{m}\frac{2}{m}\frac{2}{m}$, $\frac{4}{m}$ und $\bar{3}\frac{2}{m}$.

b) Determine the maximal subgroups of these crystal classes and the corresponding indices.

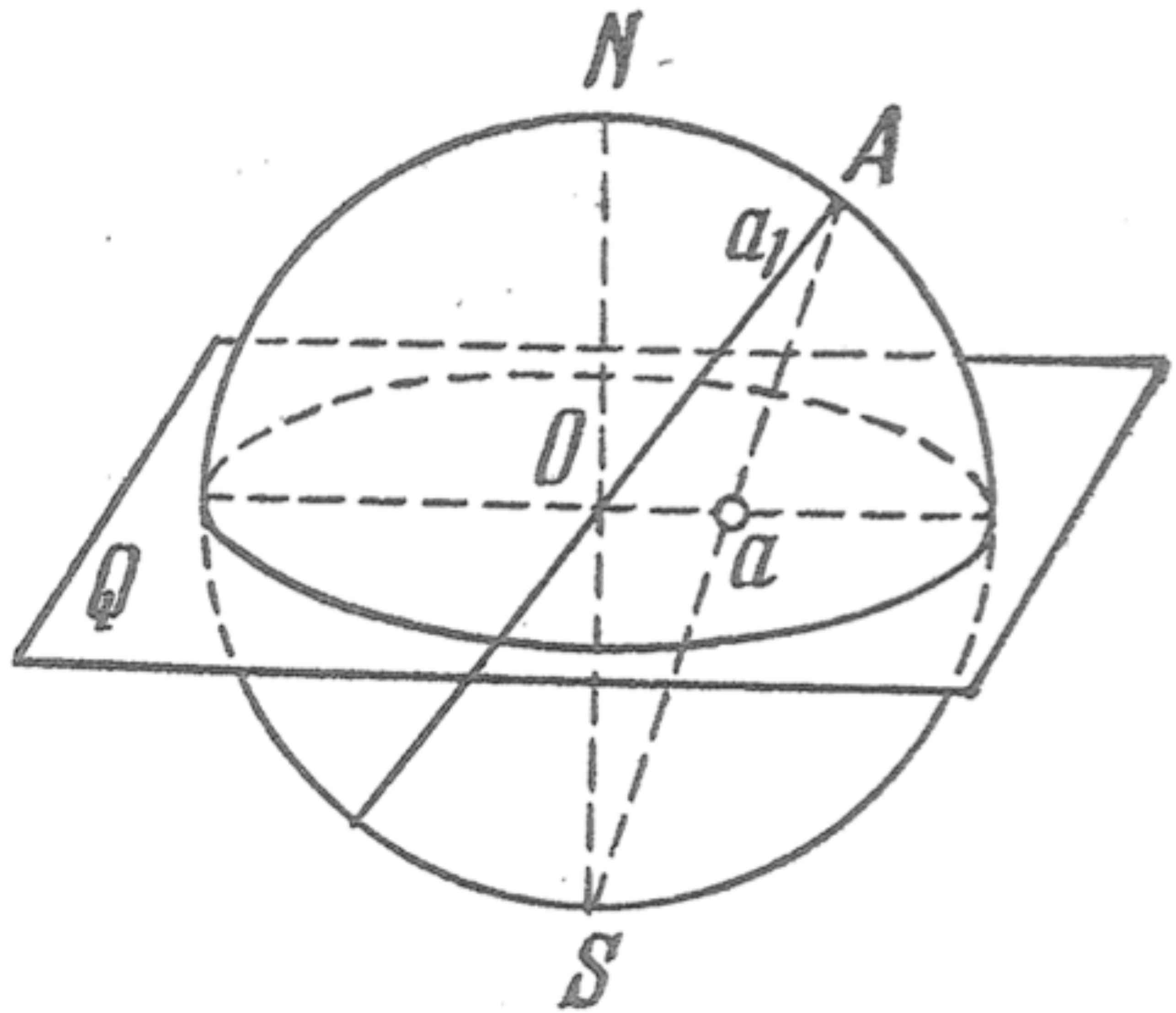
c) Which of the maximal subgroups of $\frac{2}{m}\frac{2}{m}\frac{2}{m}$ appears more than once in non-equivalent orientations?
How these subgroups are oriented with respect to the axes $\underline{a}$, $\underline{b}$ and $\underline{c}$ of the supergroup?

d) The point group $\bar{3}\frac{2}{m}$ has several equivalent subgroups of the type $\frac{2}{m}$, the so-called conjugated subgroups.
How many such subgroups are there?
Indicate the symmetry elements of the conjugated subgroups $\frac{2}{m}$ of $\bar{3}\frac{2}{m}$ on the symmetry-element stereogram of the point group $\bar{3}\frac{2}{m}$.


1
m
2
2mm
3
3m
4
4mm
6
6mm

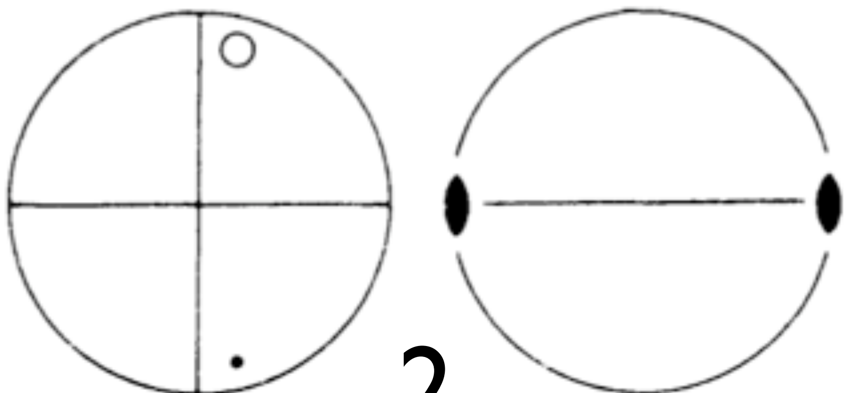
Crystallographic Point Groups

Stereographic Projections

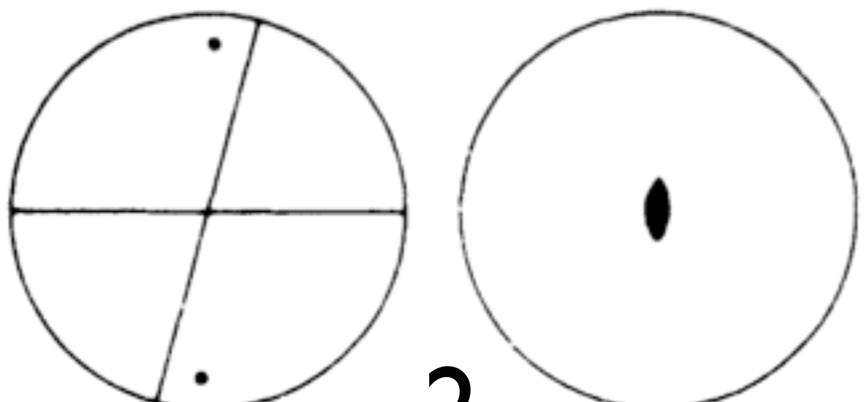


Stereographic Projections

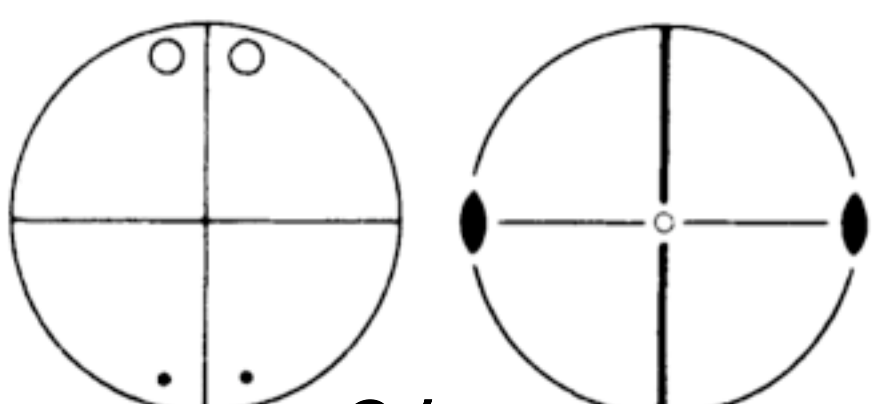
Monoclinic Point Groups



2
(unique axis b)

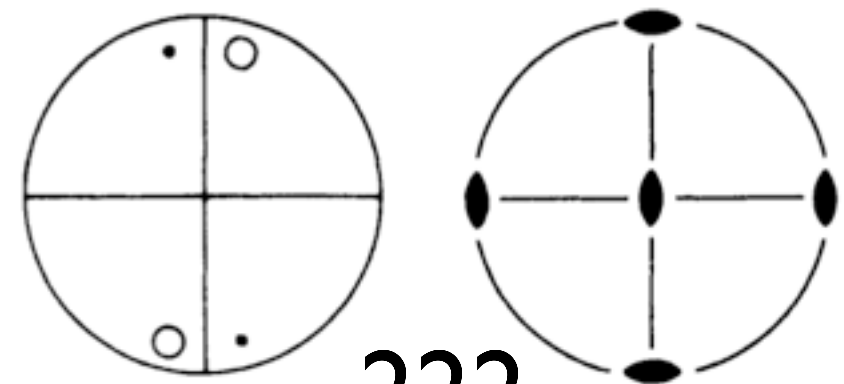


2
(unique axis c)

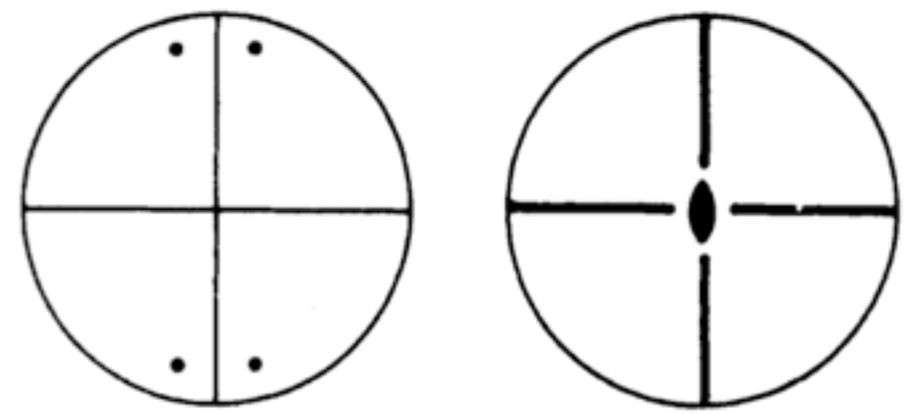


2/m

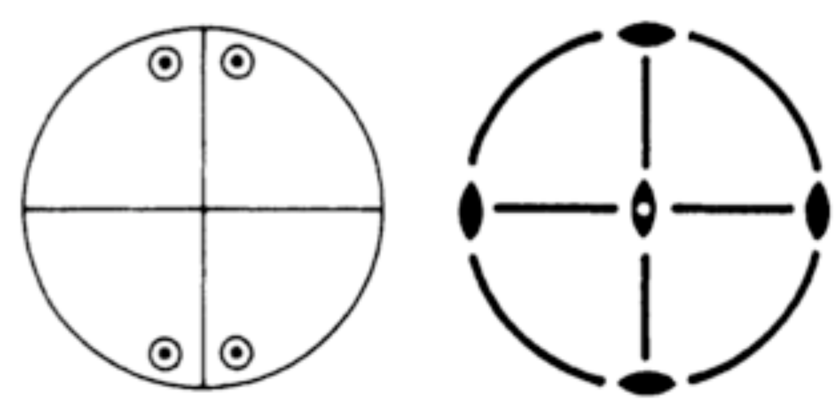
Orthorhombic Point Groups



222

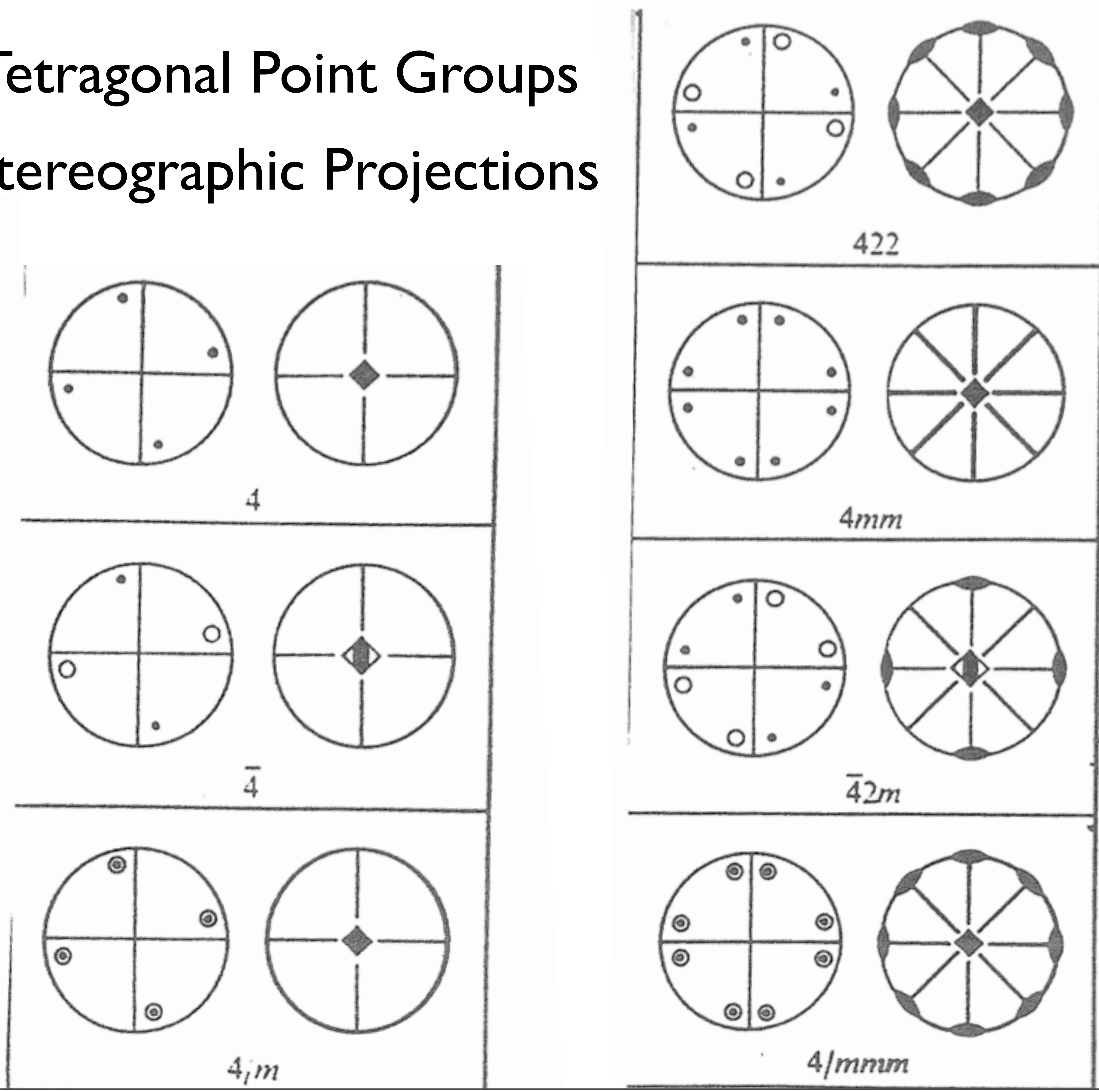


mm2



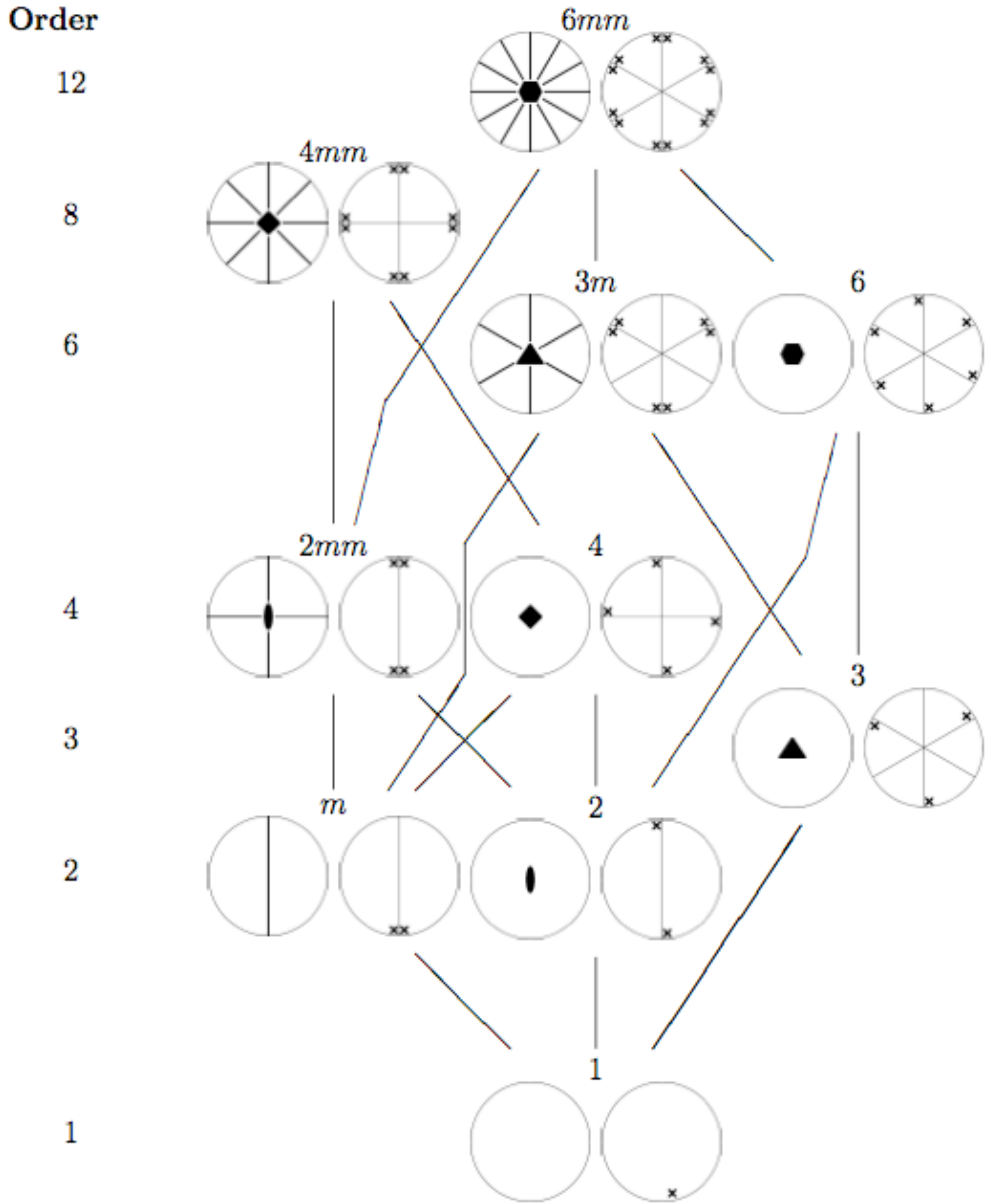
mmm

Tetragonal Point Groups
Stereographic Projections



Problem 3.5

SOLUTION



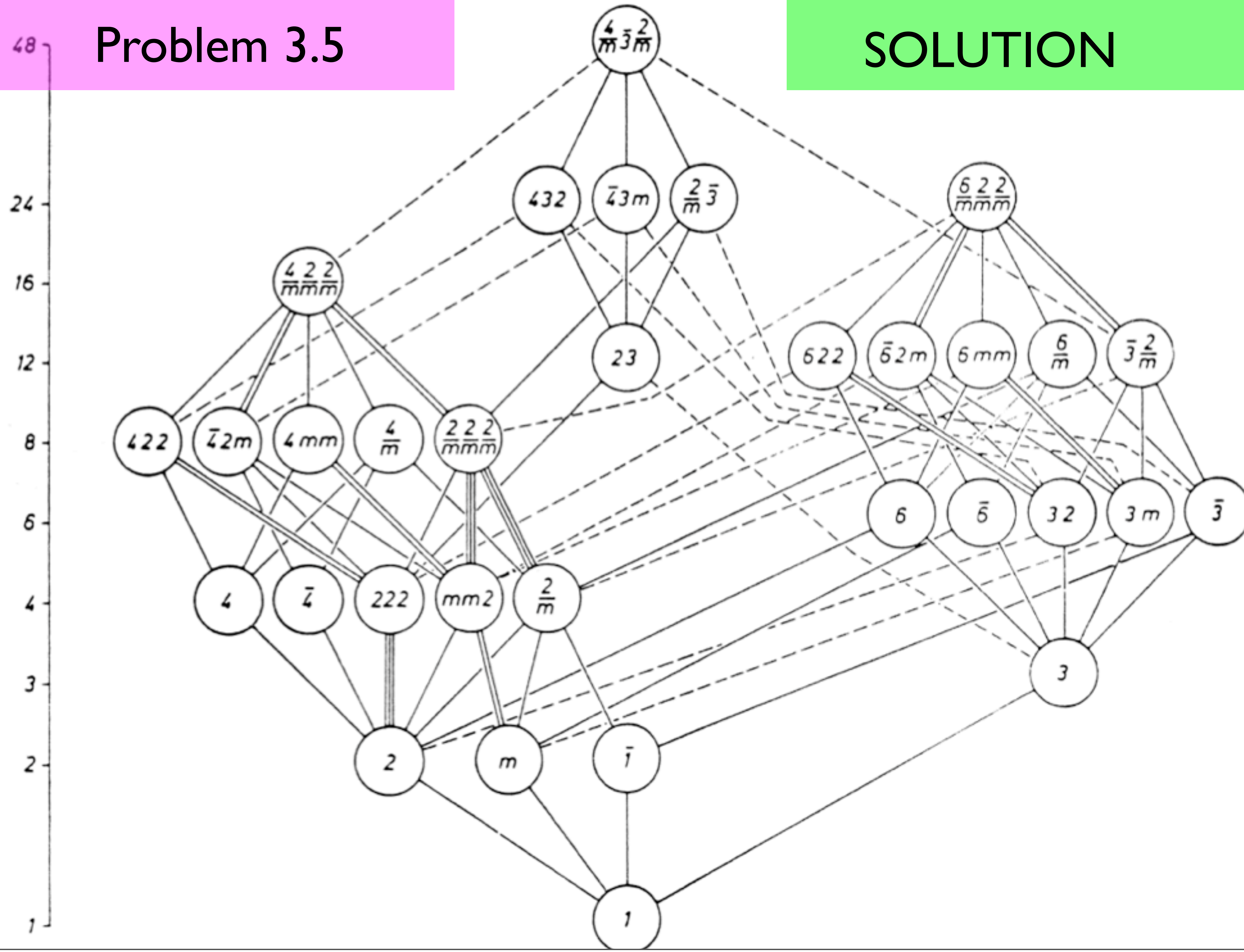
Problem 3.5

SOLUTION

(2)	**Minimal Supergroups:**	$4mm$	**Index: 2**
		$6mm$	**Index: 3**
	Maximal Subgroups:	m	**Index: 2**
		2	**Index: 2**

Problem 3.5

SOLUTION



Problem 3.5

SOLUTION

b) Maximal subgroups

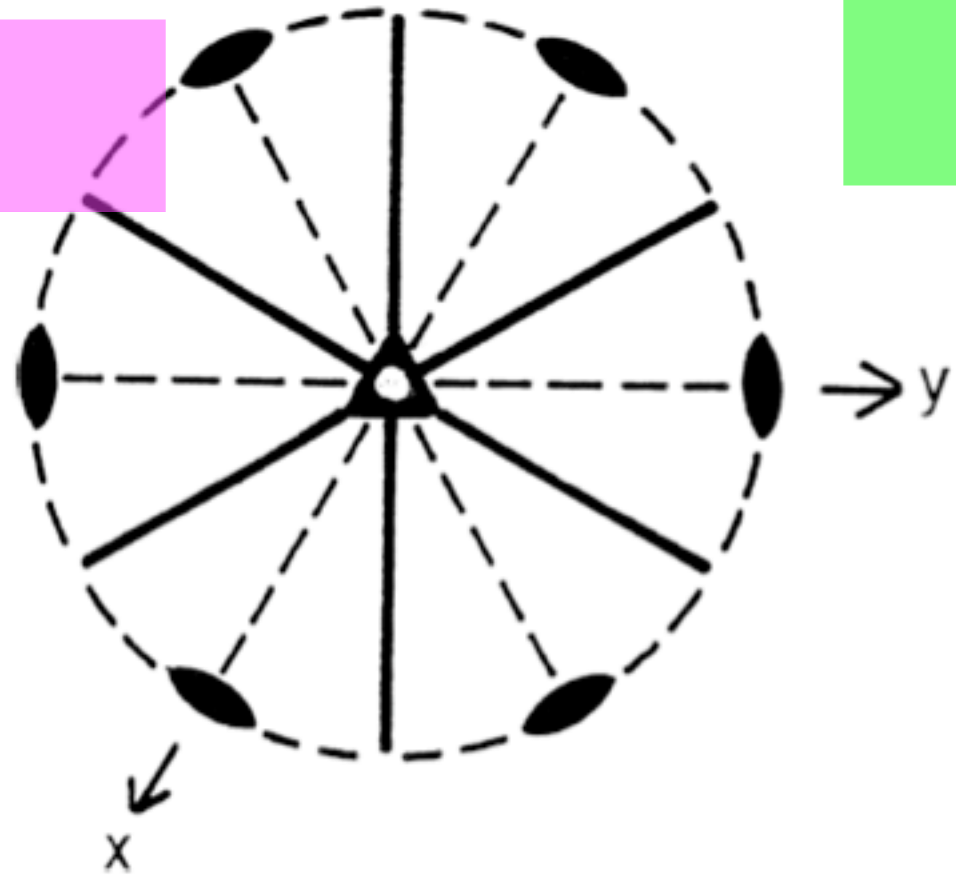
$\frac{2}{m}\frac{2}{m}\frac{2}{m}$:	222	**Index 2**
	$mm2$	**Index 2**
	$\frac{2}{m}$	**Index 2**
$\frac{4}{m}$	4	**Index 2**
	$\frac{2}{m}$	**Index 2**
	$\bar{4}$	**Index 2**
$\bar{3}\frac{2}{m}$	32	**Index 2**
	$3m$	**Index 2**
	$\bar{3}$	**Index 2**
	$\frac{2}{m}$	**Index 3**

c) $mm2$ as $2mm$, $m2m$ and $mm2$ with the two-fold axis along $\underline{a}$, $\underline{b}$ and $\underline{c}$ correspondingly.

$\frac{2}{m}$ written in full symbols $\frac{2}{m}11$, $1\frac{2}{m}1$ and $11\frac{2}{m}$ with the monoclinic axis along $\underline{a}$, $\underline{b}$ and $\underline{c}$ of the supergroup, correspondingly.

Problem 3.5

SOLUTION



Conjugate (equivalent) subgroups $\frac{2}{m}$: